Chapter 9

Finite and Edgeless

Edwin Abbott Abbott was an extremely prolific writer who produced over 45 books, and yet the novella *Flatland* is not just the only one that has seriously endured, but it is also his only work on maths; as a devout Christian, the rest were mostly on theology and literature. Thomas F Banchoff, Professor of Mathematics at Brown University since 1967, and a leading expert in *Flatland* and the geometrical and computer graphical implications of higher dimensions, writes,

> *'We may ask why the book has maintained its popularity over more than a century, and why its message about the fourth and higher dimensions has special appeal today. The answer relates to the concept of interacting with higher dimensions.'* [a]

Although *Flatland* serves as a sound basis for dimensional principles relating to the twin ideas of stacking and cross-sections, it is not on these that *Flatland 's* timelessness is based (people don't generally get excited about geometry); rather it is the deep sense within us all that there is somehow more to it than that. It is what Banchoff calls the *'method of analogy'* that excites: the potential for dimensional extrapolation with the suggestion that there might be something else out there… a clue, a hint, perhaps even a key to unlock something of the meaning behind existence… something real with which we can *'interact'*.

Computers have now been brought to bear on the visualisation of higher dimensional shapes such as the hypertorus (a 4-Dimensional donut) and 10 minutes online will whisk you away on a tour of the wonders of that of which we cannot conceive. There are many ways to engage with the subject of dimensions and geometrical shapes is one of them. As for me, my quest is to delve into the mysterious *'method of analogy'*, and to use it to pry not just into geometry, as is the custom, but perception.

To do this, the first thing we must ask ourselves is, *'What would it be like, actually living in a world of less dimensions than the one we are in?'*

The Anomaly

Flatland is a frequently referenced book – one of the reasons it is still in print – although to be fair, most writers don't go into tremendous detail, simply accepting at face value that *Sphere* who is 3-Dimensional and dwells in Spaceland represents us and the universe in which we live. This is understandable. Abbott himself has written him this way – for example in Chapter 16 of *Flatland* where he addresses us as *'Every reader in Spaceland'*.

However, to her credit, Vikki from mathematician Ian Stewart's *Flatterland* is not taken in, as she talks to her Diary about *'the days when [A Square] visited what he was TOLD was Spaceland'* and gets exasperated that *'[humans] keep changing their minds about which Space they are actually in.'* [b] Of course, the problems started with Albert Einstein's discovery that we do not inhabit a simple 3D space, but a 4D space-time. This means that in order to apply *Flatland* analogies we are going to have to get to grips with the idea of the temporal dimension. But this shouldn't be too hard since, mathematically, space-times come in any number of dimensions. Let's begin by taking a closer look at the Flatlander's viewpoint.

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[a] http://www.math.brown.edu/~banchoff/gc/ISR/ISR.html - Accessed 19th March 2016

[b] Ian Stewart, *Flatterland*, Pan Books 2003, P188

The Edge of Reality

A Square dwells in Flatland. Within our minds we cast him as an imaginary 2-Dimensional being immersed in the all-consuming flatness of his 2-Dimensional world. Ah, but were life so simple, for if we would but spend a few moments in quiet contemplation of his limited reality, we should see that all in *A Square's* world is not quite as 2D as it seems…

Certainly his world is flat, but that is not how he sees it. As Vikki realised, *A Square* views his world edge-on, 'level with the page' as it were. If you are anywhere near a table as you read this, please lower your eyes until they are level with the edge of the table (or if it's a bedside cabinet, be careful not to put your back out) and imagine there is nothing on the table except sheets of paper with drawings on them. That is how *A Square* views his world.

In Chapter 3 we condensed this phenomenon into a dimensional principle:

The 'Edge-On' Principle:

Each dimension is viewed from within itself one dimension lower.

As *Square* gazes out through his hypothetical 2D eyes, all he sees is *a line*. But a line is 1D. From his viewpoint all he sees as he turns to look around him is one continuous line which describes a full 360° circle. It's important to get this, because all that follows flows from it…

A line has only one dimension, so what *A Square* sees as he looks edge-on is 1-Dimensional. He lives in a 2D world but he *views* his entire universe in 1D. The circle it describes around him appears to him *infinite*, because a circle has no beginning and no end. However, mathematically, a circle is actually a *finite* entity because it loops back on itself. In other words, a simple geometrical circle possesses both properties: it is both finite *and* infinite when considered in different ways.

In one single defining statement we can say that the Flatlander is…

- hemmed in by a continuous circle of
  confining 1-Dimensionality with no beginning and no end.

Not only this, but as *A Square* reaches out his hand to touch his world, he also *feels* it edge-on. Indeed, in his world everything he experiences by the senses comes at him edge-on and wherever he looks he sees a line. It does not matter where he goes in his flat world there will always be an edge-on 1D line that encircles him. Like a little *Playstation* footballer, when he moves the circle moves with him, and he is always at the centre, looking out, sensing out, because he is also hearing, touching, tasting and smelling out!

Of course, because his experience of the circle is immediate, the circle can have no particular size. As a result the circle can't be thought of as existing at a certain distance from him or having a certain variable radius or depth. It's important to understand that his whole physical environment exists *at no extended distance* from him. His 1D circle is jammed right up against him, integrated into his perception. Nonetheless, for Flatlander, the edge of his 1D circle is very real.

It is his world – or as he might call it, space.

"Woops...*Aaaaaah!*"

Horror of horrors! If he can't judge distance, what if our Flatlander lives on the equivalent of a chess-board, arrives at the edge, and falls off? Fortunately for *A Square*, following on from the fact that he is hemmed in by a circle this option proves mathematically impossible. This will become clear as we go.

At this point let's recall something that EA Abbott taught us…

> *The Principle of Relationship:*
>
> **Whatever is true of the relationship between two adjacent dimensions is true of the relationship between *any* two adjacent dimensions.**

As a result, this indicates that everything that happens to *A Square* also happens to *Sphere*, except one dimension up. Just as:

- *Square* looks out from within his 2D world and sees edge-on in 1D, so
- *Sphere* looks out from within his 3D world and sees 'flat-on' in 2D.

In other words, *Sphere* sees… *flatscreen!* He lives in a 3-Dimensional universe but *his viewpoint is 2-Dimensional*. This is the *'Edge-On' Principle*[a] in action, but one dimension up.

Like a one-eyed man, *Sphere's* world, 3D though it is, is viewed by him like a 2D film – flat and without depth of field, having all the light, shade and definition of a photograph, yet *missing the ability to see objects as they exist one in front of another*, viewing them instead as moving coloured shapes. This phenomenon is not so strange as it sounds. It is actually experienced by many people throughout the world and is believed to be the result of a 'glitch' in the brain's childhood visual development. It has even been known to suddenly correct itself later in life, much to the consternation and joy of sufferers!

Flatscreen is 2-Dimensional. TVs, computers, phone-screens in our everyday world – all 2D. Although, having said that, cinema technology can fool our brains. We may be confronted with this reality when we take the kids to the pictures… *"2D or 3D – which is it to be? 2D is cheaper but 3D is (arguably) better… and did we remember the specs?"* Truth is, without the specs it's still just 2D, but blurry. Even the amazing technology of the Nintendo 3DS, which somehow managed to dispense with the glasses, succeeds by fooling the brain. The screen itself remains 2-Dimensional.

But it's hard for us to imagine how *Sphere* looks out at his 3D world and views it 'edge-on' in flatscreen, because all our 2D screens exist at an extended distance from us. Not so with *Sphere*. Just as *Square's* encompassing circle was at zero distance from his perception, so it is with *Sphere's* spherically encompassing flatscreen which is in a sense 'shrink-wrapped' around him. The reason is the same: neither of them possess visual depth.

Sphere the Square-Eyes!

So, is *Sphere* just some kind of geometrical TV addict, sitting all day glaring at the screen in front of him? How does he actually view his world? How does it appear to him?

Well, firstly, for *Sphere* perspective is not an issue, as any concept he might have of distance would be in terms of 2D, and thus might possibly be viewed isometrically. *Sphere* looks all around him like he is constantly looking around at the inside of a ball or, perhaps more accurately, a gigantic, all-encompassing

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[a] *The 'Edge-On' Principle:* Each dimension is viewed from within itself one dimension lower.

planetarium. Mathematically this is known as a 2-sphere[a], which is the boundary of a 3-Dimensional ball – like a soccer ball, or the surface of the earth – in his case viewed from the inside[b].

What *Sphere* is actually seeing is the equivalent of *A Square's* confining circle (for which the technical name is a 1-sphere), kicked up a dimension. This is the Spacelander's confining sphere.

Recalling how *A Square* was able to touch individual 2D objects edge-on, even so, *Sphere* discerns the 3-Dimensionality of his world by feeling the hypothetical 2-Dimensional surface of 'flat things' all around him. *Sphere's* reach into his world is always from the zenith down onto the flat surface before him which forms his spherical boundary – in other words, from the centre out. All this is not at all easy to picture because it is not a situation that occurs in nature and is therefore *not possible* in our world. But perhaps the most important thing to get our heads around is the idea that his 3D spherical environment exists at zero distance from him, shrink-wrapped into a 2D experience. His whole world is 'jammed up against his perception'.

We observed above that *A Square* looked out and saw himself…

- hemmed in by a continuous circle of
  confining 1-Dimensionality with no beginning and no end.

In the same way *Sphere* is…

- hemmed in by a flatscreen sphere of
  confining 2-Dimensionality with no beginning and no end.

The 2D Flatlander is surrounded by a 1D circle whilst the 3D Spacelander is hemmed in by a 2D spherical surface. This flatscreen 2-sphere upon which he is able to 'look down', and which forms the finite but edgeless boundary of 3D Spaceland, is therefore… Flatland! Flatland is a sphere, which is why – as mentioned above – *A Square* can't fall off.

Sphere gazes down upon Flatland, but his whole world turns out to be the surface[c] of a sphere. However, not in the same sense as our world, because Flatland is in fact the Spacelander's 2D field of vision – his 3D space-time *as he experiences it* – in keeping with the *'Edge-On' Principle*[d].

Reflection… This important attribute of the Flatland scenario – sphericality – was explained by the Dutch mathematician Dionys Burger in his 1965 *Flatland* update, *Sphereland*[e], written with all new characters in the same wonderful style as the original. Burger adjusts the tale's geometry in the light of Einstein's Relativity – which was still two to three decades away when *Flatland* was written – to include the key elements of curvature and expansion.

[a] An *n*-sphere is the surface or boundary of an (*n*+1)Dimensional ball.
[b] Technically the surface has no 'inner' or 'outer' because, being 2-Dimensional, it has no thickness, but he still views it from its centre. This topological acrobatic makes sense when we remember that the 2D flatscreen is at zero distance from his perception.
[c] Neither inner nor outer.
[d] *The 'Edge-On' Principle:* Each dimension is viewed from within itself one dimension lower.
[e] Dionys Burger, *Sphereland*, Harper & Row 1983

Line Dancing

This same principle also holds good down the way. The *King of Lineland* who lives one dimension down from Flatland actually inhabits *A Square's* confining circle. The *King* lives in Lineland, an endless 1D loop, and the exact same circular line that forms the boundary of *A Square's* vision.

Therefore, *A Square* gazes out upon Lineland.

When the *King*, as a hypothetical 1D being, looks out both ends of his little liney body through his (conveniently located) eyes, he sees two points, one above his head and one beneath his feet – like viewing a needle point-on. This is his view of his world looking out both ways along his 1-Dimensional line, and since a point is a 0-Dimensional entity, these twin points are referred to as 0-spheres (which is logical to mathematicians, but *shh!*, don't say anything).

Here again we see EA Abbott using his fully paid-up artistic license because, technically, *the King of Lineland would not need eyes*. His twin points are 0-Dimensional and would therefore be completely invisible to him. 0D means there's nothing to see.

All this is not at all easy for us to imagine, but if we accept *A Square's* dimensional perceptions, geometrically we must also acknowledge these as somehow also the case for the *King of Lineland* (one dimension down) and *Sphere* (one dimension up). Don't think into it *too* hard or your head will go funny! Dimensions are like that. Matt Parker, a mathematician who communicates his enthusiasm for his subject with the skill of a stand-up comedian, tells us,

> *'Accepting dimensions higher than what you can perceive takes some convincing arguments, but once you let them into your life it's a slippery uphill slope into more and more dimensions.'* [a]

Now it starts to get interesting…

Hyperland

Leaving the worthy book in 1884, I would now like to introduce you to a mythical dweller in a place I shall call Hyperland – a 4-Dimensional being who dwells a whole dimension up from *Sphere's* Spaceland, and two dimensions up from Flatland. What shall we call him? Howzabout… *Abbott!*

Abbott inhabits a 4D universe.

However, exactly like the previous characters in dimensions below him he does not actually view his world in 4D, but one dimension lower. *Abbott* looks out upon his 4D world and views it 'edge-on' in 3D (by the *'Edge-On' Principle*[b]). All around *Abbott*, the world he sees and touches is 3-Dimensional; it possesses length, width and depth.

Just as *A Square* looked out and saw himself…

- 'hemmed in by a continuous circle of
  confining 1-Dimensionality with no beginning and no end,'

and *Sphere* was…

- 'hemmed in by a flatscreen sphere of

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[a] Matt Parker, *Things to Make and Do in the Fourth Dimension*, Penguin 2015, P325-6

[b] *The 'Edge-On' Principle:* Each dimension is viewed from within itself one dimension lower.

confining 2-Dimensionality with no beginning and no end,'

even so, *Abbott* is…

- hemmed in by a depth-of-field sphere of
  confining 3-Dimensionality with no beginning and no end.

Does this sound familiar?

Abbott inhabits our world: the physical universe in which we live, which the scientists inform us is made up of three dimensions of space and one of time – our 4D cosmos. Although we can neither see nor touch time we are continually aware that it is there, marching on relentlessly to complete our 4-Dimensional experience. However, although we live in a 4D universe, the world *as we actually view it is 3D*, and it is 3-Dimensional to the touch – of these simple observations there can be no doubt. This is the expression of the *'Edge-On' Principle*[a] which applies in the real world precisely as it applies in Flatland. The 3D that we see is the confining boundary of our 4D universe which – just like *Sphere, A Square* and the *King of Lineland* – we view[b] one dimension down.

In other words, ***Flatland tells us why the world around us is 3-Dimensional***. It is precisely because the universe is 4D that we view the world in 3D.

But this carries considerable implications, because in our world our 'last' dimension, the invisible one, is time: our 'means of change'. Therefore, by applying the *Principle of Relationship*[c] we now know that the invisible last dimension in Lineland, Flatland and Spaceland must also act as the temporal dimension to each of their worlds. Earlier I quoted cosmologist Janna Levin when she wrote,

> *'As much as we try to make time the same as space, it still seems different, different enough that we continue to give it its own name. For one, we cannot move freely in time.'* [d]

The principles of *Flatland* demonstrate that time's difference from the 'spatial' dimensions is not intrinsic to time. It is not inherent within the particular character of the 4th Dimension, as scientists currently suppose, but is entirely the product of dimensional viewpoint – i.e. our location within the dimensional structure. If we were somehow able to step outside our level and view the dimensional structure of reality from 'above', what we experience as time would be seen to behave spatially. We will return to this shortly.

Deceived by Depth

It is natural for us to think of the shape and boundary of our universe in terms of how we see things, but, as with *Square* and *Sphere*, it's important (but quite hard) to understand that even for us, the boundary is not 'out there' somewhere. Consider…

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[a] *The 'Edge-On' Principle:* Each dimension is viewed from within itself one dimension lower.

[b] The wonderful design of our visual system enables us to experience 3D in a particularly dynamic way. If we cover one eye and do not move our head, we view the world in 2D – much like hypothetical *Sphere* – but because our brains are wired to process dual signals from two separate but known viewpoints, we have the impression that we can see ever so slightly round the edges of things, and by means of psychologically deciphered parallax we can accurately estimate distance and therefore speed. This enables eye-hand co-ordination, like when someone throws a ball at you and you instinctively judge the exact time and position of its arrival in your hands because your brain performs tiny lightning-fast re-calculations as it approaches. However this phenomenon is unrelated to the geometrical fact that the world is 3-Dimensional, but is in some way a product of it.

[c] *The Principle of Relationship:* Whatever is true of the relationship between two adjacent dimensions is true of the relationship between *any* two adjacent dimensions.

[d] Janna Levin, *How the Universe Got Its Spots*, Phoenix 2003

- We saw above how *Sphere* and *Square's* environments were at zero distance from them, so, by the *Principle of Relationship*[a] this is equally true for us. *3-Dimensionality exists at zero distance from us* as we look lengthwise, widthwise and depthwise all around, from and into our universe. 3D is 'jammed against our perception'. We are wholly integrated into it.
- 'Out there' for us is simply 'depth', and, like length and width it is just another of our spatial dimensions. On its own, depth as we perceive it has *nothing whatsoever* to do with the shape or location of boundaries within our universe.

Reflection… We tend to think of space and 3D as the same thing. However, that is just how we 4D dwellers happen to experience it. To be space, an extended environment does not need depth at all. Only '3D space' needs depth. The Flatlander's space is 1D and his space-time is 2D. The Spacelander's space is 2D and his space-time is 3D. Our space is…

Continuing our extrapolation and application of *Flatland* principles we will shortly look into the idea that our 'spatial' dimensions may themselves exist as a more fundamental structure to reality. The combination of length, width and depth would thus exist as a representation possessing the properties of a 3rd Dimension 'proper'.

Getting in Shape

Flatland geometry shows us our universe has a boundary. Yes, we take it for granted, but we are *part of* the very boundary of the universe – which is not someplace away out in space, but everywhere and everything you see and touch… the pen you hold, your car, the clouds, the clothes you wear, even your physical body form part of our 4D universe's 3D boundary! And like the surface of a sphere but up by one dimension, the 3-sphere boundary of our universe is finite, yet edgeless.

In addition, because we are able to extrapolate up from Flatland's 1-sphere through Spaceland's 2-sphere to Hyperland's 3-sphere, the process tells us unambiguously the shape of our universe. Much cosmological speculation is answered by the simple logic of EA Abbott's *Flatland*, which shows us that:

Our universe is geometrically equivalent to a 4-Dimensional hypersphere[b].

Or should I say… the process *confirms* to us the shape of the universe, because there is nothing scientifically new in the idea of the universe as a hypersphere. Indeed there is something of orthodoxy in this view. In Part III of his description of Relativity, Albert Einstein wrote in 1916 that,

> *'It follows from what has been said, that closed spaces without limits are conceivable. From amongst these, the spherical space (and the elliptical) excels in simplicity, since all points on it are equivalent.'* [c]

The Physicist on the popular website *Ask a Mathematician/Physicist* puts it like this,

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[a] *The Principle of Relationship:* Whatever is true of the relationship between two adjacent dimensions is true of the relationship between *any* two adjacent dimensions.

[b] The physicist's hypersphere; the mathematician's (more correct) 4-ball.

[c] Albert Einstein, *Relativity*, Routledge 2001

> *'You can think of the universe as being a 3-dimensional space that is the surface of a 4-dimensional ball, in the same way that the surface of a balloon is a 2-dimensional space wrapped around a 3-dimensional ball. … the surface of the 4-d [hyper]sphere <u>is</u> space.'* [a]

But it's wonderful to see that it was there all along, even before Relativity, nestled within the unerringly consistent principles of Edwin Abbott Abbott.

When Is a Shape Not a Shape?

Although the principles are the same for the circle and the sphere, the inference is not that the Flatlander and the Spacelander's confinements 'are' those shapes, but that they exist as the theoretical spaces which supply the environment for those lower dimensional characters *at zero distance from their perception*. Like *The Physicist* said, *'You can think of the universe as…'*

Our Earth-bound concept of 'shape' is to some extent a silly one which we need to stop imposing on reality if we want to understand it. Science writer John Gribbin explains to us that scientific models *'should always be regarded as approximations and aids to the imagination, rather than the ultimate truth.'*[b] Of course we need the concept of shape or we couldn't think or do maths! But the point I am making is that the universe-as-hypersphere is not actually what it 'is', but a useful and accurate geometrical analogy – which we will of course continue to use[c].

As we ascend the dimensions, it is this idea – of shape itself as merely a representation – that we must hold on to in order to understand how principles of geometry (of all things) might still apply in the realms of consciousness that we shall be visiting in chapters to come…

Reflection… The dimensional structure is founded on consistency, and whereas String theory may allow itself the luxury of dimensions which 'curl up' to comply with the arbitrary assumption that higher dimensions must be in some way 'spatial', all dimensions in a *Flatland*-derived framework ascend in complexity by obeying all the exact same rules. Dimensionality is essentially very simple. In fact (you may or may not be pleased to hear) it's *Reality for Dummies!*

The Missing Dimension

When *Sphere* looked down upon Flatland, he wasn't being quite the smartypants he comes across as in the book. No. All he is actually doing geometrically is looking out upon his 3-Dimensional world as he finds it, in 2D. EA Abbott writes as though *Sphere* spends his days having a 3D experience but has just chosen (nay, deigned) to look in on Flatland. But he has not. In terms of the geometry, *Sphere* ***only ever sees Flatland***. Flatland is *Sphere's* field of vision.

Reflection… But where does all this sphericality leave the idea of Flatland as a 2D *cross-section* of 3D Spaceland? Interestingly, unchanged, because both the cross-section *and* the surface of a sphere (or

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[a] http://www.askamathematician.com/2010/01/q-how-far-away-is-the-edge-of-the-universe - Accessed 29th Jan 2017

[b] John Gribbin, *The Universe: A Biography*, Penguin Science 2008, P2

[c] In Parts 5 and 6 we will examine the universe's shape in more detail, and the dimensional structure's potential to solve many current enigmas such as the smoothness of the cosmic microwave-background, the horizon problem, dark energy etc.

solid ball) are 2D, although in different ways, referred to respectively by mathematicians as Euclidean (flat) geometry and non-Euclidean (curved surface) geometry. Both hold good in geometrical analogy because they represent different aspects of the world. In terms of dimensional structure we might think of each surface as the boundary or 'container' for the next dimension up, from which cross-sections are cut. As such – and because of the subordination of the concept of 'shape' mentioned above – I will refer to them both as the book progresses.

Vive la Difference!

In conclusion, the 3D world (Spaceland) is in reality the 3-sphere boundary of the 4D world (Hyperland), which is the 4-Dimensional universe in which we live. This shows that although EA Abbott portrayed *Sphere* and Spaceland as representing his readers (human beings in the actual universe) it is clear from a closer inspection of the logic that, as confirmed earlier by mathematician Ian Stewart[a], they did not. Sorry Edwin, but they were a whole dimension too low.

Edwin Abbott Abbot grasped fully the notion of the 2D-dweller's restricted 1D field of vision (introducing the mist etc to compensate) but he failed to recognise the extrapolated principle one dimension up. He seems to have assumed that dwellers in Spaceland could actually *see* Spaceland, gazing out upon it in its entirety, which of course they can't. They only see Flatland. It is *we* who see Spaceland, because we inhabit Hyperland.

For ease of reference let's put it all together in a grid:

World	*Character*	*How they sense*	*Ds sensed*	*Ds experienced*
1D Lineland	*The King*	Point-on	0D	1D
2D Flatland	*A Square*	Edge-on	1D	2D
3D Spaceland	*Sphere*	Flat-on	2D	3D
4D Hyperland	*Abbott*	3D-on	3D	4D

But we shouldn't be too hard on the wise old Reverend. As we saw earlier, the Victorians were acutely aware of the 4th Dimension, but somewhat bemused and overwhelmed by speculations. It wasn't until 21 years after *Flatland* when Einstein cleared the air with his theory of Special Relativity that the role of time became better, though by no means completely, understood.

However, significantly, what all this demonstrates to us is that our 4th Dimension is not as *Wikipedia* and much of the scientific world would have it, *'of a different sort from the spatial dimensions'*[b], because in *Flatland* terms they are all exactly the same. 4D may appear *to us* to behave differently, but it is not of a different sort.

I feel this is worth emphasising because clearly, as with any dimension, what *Flatland* teaches us is that the 4th Dimension is inherently *neither spatial nor temporal*. All dimensions behave according to the simple and straightforward principles of *Flatland*. There is therefore no need to '*make time the same as space*', because, well, it is already. It's all just a question of viewpoint.

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[a] Ian Stewart, *Flatterland*, Pan Books 2003, P188

[b] https://en.wikipedia.org/wiki/Spacetime - Accessed 29th Dec 2012